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Opportune moment strategies for a cost spanning tree game

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Abstract Cost spanning tree problems concern the construction of a tree which provides a connection with the source for every node of the network. In this paper, we address cost sharing problems associated to these situations when the agents located at the nodes act in a non-cooperative way. A class of strategies is proposed which produce minimum cost spanning trees and, at the same time, are strong Nash equilibria for a non-cooperative game associated to the problem. They are also subgame perfect Nash equilibria.

Keywords Non-cooperative games · Spanning trees · Nash equilibria

JEL Classification C72 · D85

1 Introduction

The concept of spanning tree is of major importance for operation researchers interested in constructing network models which describe the way to connect a set of users

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to a source using the smallest amount of resources. When only the building costs are considered, the focus is on the identification of minimum cost spanning trees. The study of minimum cost spanning trees has been an important area of research, both from the theoretical point of view, where a number of efficient algorithms to produce minimum cost trees have been developed, such as Kruskal (1956) and Prim (1957), and also in applications, which have been widely used by communication companies in circuit design, cable T.V. networks, etc.

However, apart from the design of minimum cost networks, other relevant issues also arise in relation to these operation research problems. Once the tree which connects the users is obtained, if the problem of allocating the costs of the spanning tree between the users is to be considered, then the situation can be modeled as a game.

From a game theoretical perspective two kinds of situations concerning the allocation of costs in spanning trees can be explored. In a cooperative environment all the users cooperate in order to induce a consensus allocation of costs. However, the agents can also act in a non-cooperative way. In these cases the strategies they will adopt are crucial to the outcome.

The cooperative spanning tree model is well-known from Bird's paper (1976). He proposed a cost allocation scheme that consists of assigning to each user the cost of the incident edge in the unique path that links this user with the source. Granot and Huberman (1981) then showed that the allocations arising from a Bird's cost allocation scheme are always in the core of the minimum cost spanning tree cooperative game. A survey of cooperative games associated to cost spanning tree problems can be seen in Borm et al. (2001).

However, in some situations where the focus is on the analysis of the stability and efficiency of a social and economic model of minimum cost spanning tree formation, a non-cooperative approach for the allocation of costs may be appropriate. Several papers study cost spanning tree problems from this point of view. Mutuswami and Winter (2002) introduce a mechanism for network formation in a framework where agents have some private benefits which affect the final outcome. However, costs are usually observable whilst benefits are often not, and in the classic setting where private benefits are not taken into account, their mechanism is not easy to interpret. For instance, as already pointed out by Bergantiños and Lorenzo (2004), the role of the source node is unclear.

The analysis of minimum cost spanning tree formation presented in our paper is based on the non-cooperative multi-stage game introduced in Bergantiños and Lorenzo (2004). In this approach (as in others already used in the literature; see for instance Bergantiños and Lorenzo (2004); Bergantiños and Lorenzo (2005), and the references therein) we assume that agents only play pure strategies and agents' decisions depend only on who is already connected to the network and not on the order in which agents did connect, i.e, one assumes that past history may not be available and therefore, decisions are made based on current information. Moreover, in our proposal a minimum cost spanning tree has to be constructed for the agents who are involved up to any stage. This is a collective efficiency property that should be required when modeling a variety of situations, for instance, in those cases in which a central authority is funding the project and will not support it unless this condition is fulfilled.

Our interest lies in the identification of those strategies that produce minimum cost spanning trees which are at the same time Nash equilibria.

Bergantiños and Lorenzo (2004) study a class of strategies for these games which are Nash equilibria but do not necessarily yield minimum cost spanning trees. These authors also, in Bergantiños and Lorenzo (2005), study another class of strategies in which the agents have threshold costs, and prove that they are optimal provided that they are Nash equilibria. Note that these strategies need not even be Nash equilibria.

In this paper, we introduce the class of *opportune moment strategies* which, at the same time, are Nash equilibria and produce minimum cost spanning trees. We also prove that these strategies are subgame perfect Nash equilibria and strong Nash equilibria.

In addition, it is also important to note that the payoffs provided by any profile of opportune moment strategies constitute a core cost allocation in corresponding the cooperative game, since they are Bird cost allocations. Although there is a well-known criticism of Bird tree allocations (Granot and Huberman 1981), in that it discriminates against players that connect earlier to the source, this criticism is not relevant in our non-cooperative framework. Indeed, the chance of ending up unconnected (and thus paying the penalty cost) does not increase with an earlier connection, and in most cases decreases. Hence, our opportune moment strategies can be seen as a new justification for the usage of Bird allocations in minimum cost spanning tree games.

The paper is organized as follows. Basic concepts on minimum cost spanning trees are provided in Sect. 2, where a result, on which part of our analysis relies, is also presented. In Sect. 3, a non-cooperative game that models how the agents connect to the source is described. Finally, the class of opportune moment strategies is proposed and analyzed in Sect. 4.

2 Basic concepts

There is a finite set of nodes, $N = \{1, 2, \ldots, n\}$, and each of them has to connect to a common root, 0, the source node or the common supplier.

For any subset of nodes $S \subseteq N$, denote by $S_0 = S \cup \{0\}$ and by G_S the *graph* (S_0, E_{S_0}), where S_0 represents the set of *nodes* and E_{S_0} is the set of *arcs or edges*, $E_{S_0} \subseteq \{(i, j) \in S_0 \times S_0, \mid i \neq j\}$. The element $(i, j) \in E_{S_0}$ is an edge that connects node i and node j. The graph G_S is said to be a *complete graph* if $E_{S_0} = \{(i, j) \in S_0 \times S_0, \mid i \neq j\}$.

Given G_S, for $i, j \in S_0$, a subset $P(i, j) \subseteq E_{S_0}$ is a *simple path* (a path from now on) from i to j if $P(i, j) = \{(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_{k+1})\}$, where $i_1 = i, i_{k+1} = j$, with $k \geq 1$ and the intermediate nodes, $i_2, i_3, \ldots, i_k$, are all different from i and j and appear exactly twice in the sequence. A *cycle* is a path whose initial and final nodes coincide.

The graph G_S is said to be *connected* if for each $i, j \in S_0, i \neq j$, there is at least a path from i to j. A connected graph, G_S, that does not contain any cycle is said to be a *spanning tree* on S_0. To simplify the notation we denote by T_S the spanning tree on S_0 and also the set of edges in the tree. Let $\mathcal{T}^S$ denote the set of all spanning trees on S_0.

Let G_N be the complete graph (N_0, E_{N_0}). Denote by c_{ij} the cost associated with $(i, j) \in E_{N_0}$, and by c the matrix of costs $c = (c_{ij})_{i,j \in N_0}$. We assume that $c_{ij} \geq 0$, $c_{ij} = c_{ji}$ for all $i, j \in N_0$ and $c_{ii} = 0$ for all $i \in N_0$. The total cost of a spanning tree, T_S, is $c(T_S) = \sum_{(i,j)\in T_S} c_{ij}$. A spanning tree, T_S, is a *minimum cost spanning tree* (MCST from now on) if $T_S \in arg \min_{T \in \mathcal{T}^S} c(T)$.

For $i \in N \backslash S$, denote by $m_i^S = \min_{j \in S_0}\{c_{ij}\}$ the minimum connection cost of node i to a node in S_0, and by $S_0^i = \{j \in S_0 \mid c_{ij} = m_i^S\}$ the set of nodes in S_0 to which node i can connect at cost m_i^S. Let m^S be the minimum connection cost of nodes in $N \backslash S$ to nodes in S_0, $m^S = \min_{i \notin S}\{m_i^S\}$, and let M^S be the set of nodes that can connect to S_0 with this minimum cost, $M^S = \{i \in N \backslash S \,|\, \exists\ j \in S_0,\ c_{ij} = m^S\}$.

For each $i \in N$ denote by B^i the set of nodes to whom node i could connect at its minimum cost, $B^i = \{j \in N_0 \backslash \{i\} \mid c_{ij} \leq c_{ik},\ \forall k \in N_0 \backslash \{i\}\}$. Let M_*^S be the set of nodes in $N \backslash S$ that can connect to S_0 with their cheapest connection, $M_*^S = \{i \in N \backslash S \mid B^i \cap S_0 \neq \emptyset\}$. For each element $k \in M_*^S$ we associate (arbitrarily) a *unique* element $l(k) \in B^k \cap S_0$. Then we set $E_*^S = \{(k, l(k)) | \, k \in M_*^S\}$.

We now establish a result about minimum cost spanning trees that will be instrumental in the analysis of the problem presented in this paper.

Theorem 2.1 *Let T_S be a spanning tree on $S_0 \subset N_0$, that is part of some MCST on N_0. If $i \in M^S$, $j \in S_0^i$, then, at least one of the MCSTs on N_0 contains $T_S \cup \{(i, j)\} \cup E_*^S$.*

Proof As a consequence of the properties on spanning trees, (see, for instance, Theorem 2.1, page 11, in Wu and Chao (2004)) if, when $i \in M^S$ and $j \in S_0^i$, the set of trees (the forest) $\{T_S, T_{\{i\}},\ i \in N \backslash S\}$ is considered, then at least one of the MCSTs on N_0 contains $T_S \cup \{(i, j)\}$ since c_{ij} is the cheapest connection between T_S and any other tree in the forest. Now, by considering the forest $\{T_S \cup (i, j), T_{\{k\}},\ k \in N \backslash S \cup \{i\}\}$ and any $i^* \in M_*^S$, the same reasoning can be applied to connect i^* to a unique node $j^* \in S_0^{i^*}$, since $c_{i^* j^*}$ is the cheapest connection for i^* and therefore is the cheapest connection between $T_{\{i^*\}}$ and any other tree in the forest. Hence, at least one of the MCSTs on N_0 contains $T_S \cup \{(i, j) \cup (i^*, j^*)\}$. All the remaining nodes $i^* \in M_*^S$ can be connected to a node in S_0 in the same way and the result follows. □

It is important to point out that to derive the result in Theorem 2.1 it is not sufficient for T_S to be a MCST on S_0, but it is necessary that T_S be contained in a MCST on N_0. The following example illustrates this point.

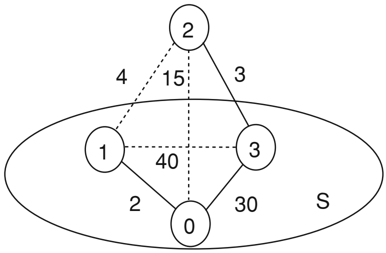
Example 1 Consider $N = \{1, 2, 3\}$ and the complete graph on N_0 represented in Fig. 1, where the connection costs are $c_{01} = 2$, $c_{02} = 15$, $c_{03} = 30$, $c_{12} = 4$, $c_{13} = 40$, $c_{23} = 3$.

For $S = \{1, 3\}$ the tree $T_S = \{(0, 1), (0, 3)\}$ is a MCST on S_0. However, when agent 2 uses his cheapest connection $T_S \cup \{(2, 3)\}$ is not a MCST on N_0.

3 A non-cooperative cost spanning tree game

The cooperative MCST game arises when analyzing the problem of allocating the costs associated to a spanning tree in a graph between the users or agents which are

Fig. 1 Illustration of Example 1



located at the nodes of a graph, with a node reserved for a common supplier with no participation in the cost sharing problem. This model has been widely studied in the literature (Bird 1976; Granot and Huberman 1981; Borm et al. 2001). One relevant result on this topic is that allocations of costs arising from Bird's cost allocation rule are always in the core of the game.

However, in many real-life situations, agents make their decisions independently and therefore it is convenient to address the problem from a non-cooperative point of view. The following is the description of a non-cooperative game representing the network formation in a cost spanning tree problem. A reason why the agents should join such a game could be that some authority (it might be the grand coalition itself) chooses this mechanism as a tool to decide which edges to build and how to allocate the costs.

Consider the set of players (or agents) $N = \{1, 2, \dots, n\}$. Each player has to connect his node to the common supplier. In order to describe how are the agents going to make the connections to the source, a non-cooperative multi-stage game, (N_0, c), is associated to each cost spanning tree problem in the complete graph G_N with costs c.

Initially, no edges are constructed and all the nodes in G_N are unconnected. At the first stage each agent decides whether to connect to the source or not. If no agent connects or every agent connects, then the game finishes. Otherwise, the game proceeds to a second stage. In subsequent stages non-connected agents face a set of agents that are already connected, and have to decide whether to remain unconnected or to connect to one of the connected agents or directly to the source. The game finishes at the stage when no more agents connect or when all the agents are already connected.

In this approach we assume that agents only play pure strategies and agents' decisions depend only on who is already connected to the network and not on the order in which agents have connected. In other words, the decisions of the agents depend only on what "is on the table" and not on the way in which things are "put on the table".

At any stage, if agent i connects to agent j, then agent i pays the connection cost c_{ij}. It is assumed that all the agents want to be connected, even if they have to pay their highest cost, since otherwise they would not join the game. This situation can be formalized by stating that if when the game finishes some agent is not connected, then he pays a very high penalty cost as compared with his most expensive connection cost.

Denote by $2_0^N(i)$, the set of all the coalitions which contain the source and do not contain agent i, $2_0^N(i) = \{S_0 = S \cup \{0\}, \mid i \notin S \subset N\}$. Since the decision of each agent depends only on the set of agents already connected, a strategy for agent $i \in N$

is a map $x_i : 2_0^N(i) \longrightarrow N_0 \cup \{a\}$, such that $x_i(S_0) \in S_0 \cup \{a\}$, where $x_i(S_0) = j \in S_0$ means that agent i connects to agent j, and $x_i(S_0) = a$ means that agent i remains unconnected.

Any profile of strategies of the set of agents, N, is denoted by $x = (x_i)_{i \in N}$. Denote by X_i the set of all possible strategies for agent i and by X the set of all possible profiles of strategies of the set of agents, N.

A profile of strategies x of the game (N_0, c) induces a graph which is a tree on a subset $S_0 \subseteq N_0$, denoted by T^x. This tree is not necessarily a spanning tree on N_0. In contrast to the analysis in Bergantiños and Lorenzo (2004); Bergantiños and Lorenzo (2005), we assume that a MCST for the subset of connected agents has to be built at every stage of the game. If the total cost paid by all the players is to be minimized, then the solution should consist of a spanning tree, since unconnected agents will have to pay a very large cost.

In what follows our interest is focused on the characterization of those profiles of strategies that are Nash equilibria and at the same time induce a MCST. These two conditions represent first, the individual rationality requirement which underlies the concept of Nash equilibria, and secondly, a collective rationality property, which indicates that it is not possible to improve the final outcome of the game in a collective sense.

In order to formally define these properties we use the following notation. Given a profile of strategies x for the whole set of players, N, and a subset of players $S \subset N$, denote by x_S (x_{-S}) the projection of x on S $(N \setminus S)$ that represents the corresponding profile of strategies for the agents in S $(N \setminus S)$. By $(x; x'_S)$ we represent the profile of strategies in which agents in S deviate from x by using the profile of strategies x', that is, $(x; x'_S)_i = x'_i$ for $i \in S$ and $(x; x'_S)_j = x_j$ for all $j \notin S$.

Let $c_i(x)$ denote the connection cost for agent i when a profile of strategies x is adopted. The total cost induced by x is denoted[1] by $c(x)$, $c(x) = \sum_{i \in N} c_i(x)$.

Definition 3.1 The profile of strategies $x \in X$ is a *Nash equilibrium* (NE) for the game (N_0, c), if for every agent $i \in N$, $c_i(x) \leq c_i(x; x'_i)$ for all $x'_i \in X_i$, $x'_i \neq x_i$.

That is, x is a NE if any unilateral deviation of agent i from the profile of strategies x does not yield an improvement in the cost assigned to agent i.

Notice that in the setting of this paper, any NE induce a spanning tree on N_0.

Definition 3.2 The profile of strategies $x \in X$ is a *strong Nash equilibrium* (SNE) for the game (N_0, c), if for each coalition $S \subset N$ and each $x'_S \neq x_S$, there is at least one agent $i \in S$ such that $c_i(x) \leq c_i(x; x'_S)$.

It is easy to see that, in general, this last equilibrium concept is stronger than that of NE. A SNE is such that deviations from the strategy of any group of agents will not produce an improvement in the cost of all the agents that deviate.

Nevertheless, in our framework, a NE of the game (N_0, c) is also a SNE, as is stated in the following Lemma.

[1] We will slightly abuse notation by using $c_i(x)$ and $c(x)$ instead of $c_i(T^x)$ and $c(T^x)$.

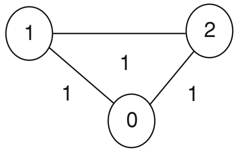


Fig. 2 Illustration of Example 2

Lemma 3.3 *If $x \in X$ is a Nash equilibrium for the game (N_0, c), then x is a strong Nash equilibrium.*

Proof Suppose on the contrary that x is a non-strong Nash equilibrium for a game (N_0, c). Let $H \subset N$ be a subset of agents and y be a profile of strategies of the agents in H such that $c_i(x; y_H) < c_i(x)$ for all $i \in H$. Let $T \subseteq N \setminus H$ be the set of players who connect independently of the actions of the members of H, i.e., who connect if the members of $N \setminus H$ play according to x and if agents in H always play a. Hence, for all $i \in H$, $c_i(x) \leq \min_{j \in T \cup \{0\}}\{c_{ij}\}$, since x is a NE and the strategy of waiting until all members of T are connected costs at most this amount. Let i be the first player in H that connects when $(x; y_H)$ is played. Then i uses an arc to a node in $T \cup \{0\}$, and therefore $c_i(x; y_H) \geq \min_{j \in T \cup \{0\}}\{c_{ij}\}$. This is a contradiction. □

Let $x = (x_i)_{i \in N}$ be a profile of strategies for the set of agents, N, and let T^x be the tree induced on $S'_0 \subseteq N_0$ (this tree connects a set of agents S' to the source). Let T_S be any subtree of T^x, which contains the source node, where $S \subset S'$ is the set of involved agents.

Denote $N \setminus S$ by $\bar{S}$ and shrink T_S to a fictitious node $\tilde{0}$ that will be considered the source for the nodes in $\bar{S}$. We define the subgame $(\bar{S}_{\tilde{0}}, c^S)$, where $\bar{S}_{\tilde{0}} = \bar{S} \cup \{\tilde{0}\}$ and $c^S_{ij} = c_{ij}$ for all $i, j \in \bar{S}$ $c^S_{i\tilde{0}} = \min_{k \in S_0}\{c_{ik}\}$ for all $i \in \bar{S}$.

Definition 3.4 The profile of strategies $x \in X$ is a *subgame perfect Nash equilibrium* (SPNE)[2] for the game (N_0, c), if x_{-S} is a NE for any subgame $(\bar{S}_{\hat{0}}, c^S)$.

Note that not all NE are SPNE, as is shown in the following example (see Fig. 2).

Example 2 Consider a graph where two players have the same connection costs with the source and between them, that is, $c_{01} = c_{02} = c_{12} = 1$.
Let x be defined by $x_i(\{0\}) = 0$ and $x_i(\{0, 1, 2\} \setminus \{i\}) = a$. Then x is a non-subgame perfect Nash equilibrium, because the players do not play optimally in the one-player subgame.

In general, a NE does not induce a MCST. Moreover, there exist profiles of strategies which are not NE, but induce MCST on N_0. This is shown in the following example.

Example 3 Consider $N = \{1, 2, 3\}$ and the complete graph on N_0 with connection costs $c_{01} = 10$, $c_{02} = 25$, $c_{03} = 15$, $c_{12} = 20$, $c_{13} = 15$, $c_{23} = 5$. Figure 3 shows the MCSTs $T = \{(0, 1), (1, 3), (2, 3)\}$ and $T' = \{(0, 1), (0, 3), (2, 3)\}$ with total cost $c(T) = c(T') = 30$, on N_0.

[2] See Van Damme (1991).

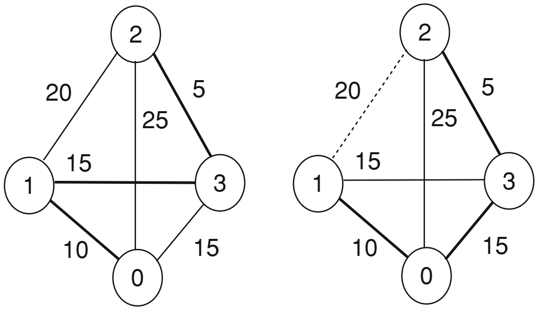


Fig. 3 T and T' are MCSTs on the complete graph

Consider the profile of strategies $x = (x_i)_{i \in N}$, defined as follows:

$$x_i(S_0) = \begin{cases} j \in S_0^i & \text{if} \quad i-1 \in S_0 \\ a & \text{otherwise} \end{cases}$$

This strategy is a NE but it generates the tree $T^x = \{(0,1),(1,2),(2,3)\}$ which is not a MCST, since $c(x) = 35$.

On the other hand, consider the profile $y = (y_i)_{i \in N}$, where

$$\begin{array}{llll} y_1(\{0\}) = 0 & y_1(\{0,3\}) = 0 & y_1(\{0,2\}) = 0 & y_1(\{0,2,3\}) = a \\ y_2(\{0\}) = a & y_2(\{0,1\}) = 1 & y_2(\{0,3\}) = 3 & y_2(\{0,1,3\}) = 3 \\ y_3(\{0\}) = 0 & y_3(\{0,1\}) = a & y_3(\{0,2\}) = 2 & y_3(\{0,1,2\}) = 2 \end{array}$$

The profile y produces the minimum cost spanning tree $T^y = T'$. However, it is not a NE, since if agent 3 deviates by adopting the following strategy

$$y_3'(\{0\}) = a \quad y_3'(\{0,1\}) = a \quad y_3'(\{0,2\}) = 2 \quad y_3'(\{0,1,2\}) = 2$$

then the tree generated is $T^{(y;y_3')} = T = \{(0,1),(1,2),(2,3)\}$, and agent 3 pays a lower cost, since $c_3(y;y_3') = 5$, while $c_3(y) = 15$.

4 Opportune moment strategies

The idea underlying the strategies described below is that each agent would like to connect using his cheapest connection. In the case where each agent's cheapest connection generates a spanning tree, this tree is a MCST. However, in general this strategy does not induce a spanning tree and further analysis is necessary to identify the strategies that the agents will adopt in the game. A first observation is that when at a certain stage of the game (N_0, c) the cheapest connection of an agent is feasible, he will connect to the existing tree, by using what we call his *best individual opportunity*, but when this is not the case, he would like to delay the connection, since by waiting he may have the opportunity of a better connection.

Nevertheless, the necessity of producing minimum cost spanning trees forces the agents to connect at a certain moment and use what we call their *best collective* *opportunity*. That is, each agent will have to connect at the stage in which he provides the cheapest connection from among all the feasible connections.

We incorporate a ranking of the players in order to solve the ties when there are several agents that can use their *best collective opportunity*. Let $\sigma : N \to N$ denote a permutation function that reflects this ranking. For each ranking, σ, let $f_\sigma : 2^N \to N$ denote the function that assigns to each coalition $M \in 2^N$ the agent, $i_\sigma \in M$ ($f_\sigma(M) = i_\sigma \in M$), such that $\sigma(i_\sigma) \leq \sigma(i)$, $\forall i \in M$.

The profiles of strategies in the following class depend on the ranking, σ, used as a tie breaker to select the agent who is going to connect in the present stage from among those who provide the cheapest connection to the existing tree. That is, given the ranking, σ, when agents in S are already connected, a unique agent in $M^S = \{i \in N \backslash S \mid \exists\, j \in S_0,\ c_{ij} = m^S\}$, $f_\sigma(M^S)$, is chosen to connect to the existing tree. These profiles of strategies include also the possibility of connecting by using the agents' best individual opportunity. We will call these profiles *profiles of opportune moment strategies*.

Definition 4.1 Given a ranking of the players, σ, a profile of opportune moment strategies, $x^\sigma \in X$, for the game (N_0, c) is defined as follows:

$$x_i^\sigma(S_0) = \begin{cases} k \in B^i \cap S_0 & \text{if} \quad B^i \cap S_0 \neq \emptyset \\ j \in S_0^i & \text{if} \quad i \in M^S \text{ and } f_\sigma(M^S) = i \\ a & \text{otherwise} \end{cases}$$

Notice that by using a profile of opportune moment strategies, more than one agent may connect to the existing tree at any stage, but only one of these agents uses his best collective opportunity. Furthermore, a profile of opportune moment strategies is well defined since, if at the same time, $B^i \cap S_0 \neq \emptyset$ and $i \in M^S$, then $B^i \cap S_0 = S_0^i$.

Even for a fixed ranking of the players, the profile of opportune moment strategies may not be unique. If for agent i, who is going to connect to the existing tree, S_0^i or $B^i \cap S_0$ are not singletons, any choice leads to different spanning trees with the same costs for the agent.

It is important to point out that when the costs associated to the connections in the network are all different, a ranking of the players is not needed since no ties occur and a unique MCST exists. In this case every ranking induces the same strategy.

Notice also that opportune moment strategies differ from those strategies inspired by Prim's algorithm (i.e., those strategies which consist of connecting to the tree at the stage when the agents provide the cheapest connection to the existing tree) with a ranking as a tie breaker, that is:

$$y_i^\sigma(S_0) = \begin{cases} j \in S_0^i & \text{if} \quad i \in M^S \text{ and } f_\sigma(M^S) = i \\ a & \text{otherwise} \end{cases} \tag{4.1}$$

It is easy to see that, given a ranking of the players, σ, if x^σ is a profile of opportune moment strategies and $y^\sigma = (y_i^\sigma)_{i \in N}$ is the above strategy, then T^{x^σ} does not necessarily coincide with T^{y^σ}, as can be seen in the following example.

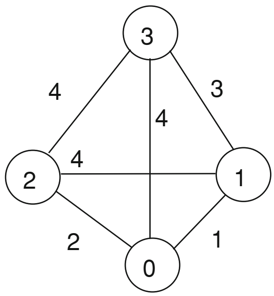


Fig. 4 Complete graph (Example 5)

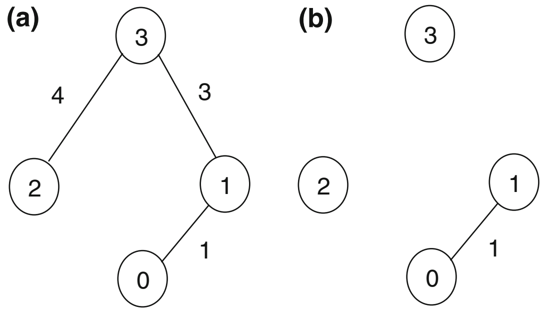


Fig. 5 Illustration of Example 5

Example 4 Consider the graph in Example 2. There are two possible rankings of the players, σ_1 and σ_2, where $\sigma_1(1) = 1$, $\sigma_1(2) = 2$ and $\sigma_2(1) = 2$, $\sigma_2(2) = 1$. f_{σ_1} generates two spanning trees by using y^{σ_1}: $T_1^{y^{\sigma_1}} = \{(0,1),\ (1,2)\}$ and $T_2^{y^{\sigma_1}} = \{(0,1),\ (0,2)\}$. Analogously, $T_1^{y^{\sigma_2}} = \{(0,2),\ (1,2)\}$ and $T_2^{y^{\sigma_2}} = \{(0,1),\ (0,2)\} = T_2^{y^{\sigma_1}}$ are the two spanning trees generated by using y^{σ_2}. Nevertheless, if the players adopt a profile of opportune moment strategies, then only one spanning tree is generated regardless of the ranking considered, $T^{x^{\sigma_1}} = T^{x^{\sigma_2}} = \{(0,1),\ (0,2)\}$, and the game finishes at the first stage.

Moreover, for an agent i it is always at least as good to choose a profile of opportune moment strategies, x_i^{σ}, as to play y_i^{σ}. The following example shows that the improvement for that agent i can be strict.

Example 5 Consider the graph with costs represented in Fig. 4.
The strategy used by player 1 is y_1^{σ} and the strategy of player 2 consists of connecting to player 3 when player 3 is connected, and remaining unconnected meanwhile.

If player 3 uses a profile of opportune moment strategies, then the tree in Fig. 5a is constructed and he pays a connection cost of 3. However, if player 3 uses y_3^{σ} he will be unconnected when the game ends, as represented in Fig. 5b, and therefore, he will have to pay the penalty cost.

This result is formally established in the following theorem.

Theorem 4.2 *Let σ be a ranking for the players. For all $i \in N$, let x_i^{σ} be an opportune moment strategy for player i and y_i^{σ} be the strategy given in 4.1. Then $c_i(x; x_i^{\sigma}) \leq c_i(x; y_i^{\sigma})$ for all $x \in X$.*

Proof Let $x \in X$ be a profile of strategies for the set of players N. If, when player i deviates from x by using x_i^σ, the game ends and player i is still unconnected, then he would also be unconnected if he had used y_i^σ, and the result follows. On the other hand, by deviating from x by using x_i^σ, player i connects when a group of players $S \subset N$ is already connected. In this case $c_i(x; x_i^\sigma) = c_{ij}$, for some $j \in S_0$. If $j \in B^i \cap S_0$, then $c_{ij} = c_i(x; x_i^\sigma) \leq c_i(x; y_i^\sigma)$, since c_{ij} is the minimum cost that player i can pay. Otherwise, $j \in S_0^i$, and $c_{ij} = c_i(x; x_i^\sigma) = c_i(x; y_i^\sigma)$ since, by deviating from x by using y_i^σ, player i connects to $j' \in S_0^i$ and $c_{ij} = c_{ij'} = m^S$. □

The following result states that a profile of opportune moment strategies is a SPNE and induces a MCST on N_0.

Theorem 4.3 *Given a ranking for the players, σ, a profile of opportune moment strategies for the game (N_0, c) induces a MCST on N_0 and is a SPNE. Conversely, if the profile of strategies, x, induces a MCST on N_0, then a ranking of the players, σ, and a profile of opportune moment strategies, x^σ, exist, such that $c_i(x) = c_i(x^\sigma)$ for all $i \in N$.*

Proof By recursively applying the result stated in Theorem 2.1, it follows that the tree generated by a profile of opportune moment strategies is a MCST on N_0.
To prove that a profile of opportune moment strategies, x^σ, is a SPNE consider the spanning tree T^{x^σ}, any subtree T_S of T^{x^σ}, with root on the source node (which could possibly consist of only the source node), and the set of agents $S \subset N$ involved ($S_0 = S \cup \{0\}$). Let $(\bar{S}_{\hat{0}}, c^S)$ be the corresponding subgame. Let us assume that player $i \notin S$ deviates from x^σ by using x_i. One of the following situations occurs:

1. Either
   (a) $B^i \cap S_0 = \emptyset$ and $i \notin M^S$, or
   (b) $B^i \cap S_0 = \emptyset$, $i \in M^S$ and $f_\sigma(M^S) \neq i$.

   If player i deviates from x^σ by using x_i, which consists of connecting to any node $j \in S_0$ of the current subtree, then $(x^\sigma; x_i)_i(S_0) = x_i(S_0) = j \in S_0$ and player i pays $c_i(x^\sigma; x_i) = c_{ij} \geq m_i^S$. Since every player in $\bar{S}$, except for player i, uses $x^\sigma_{-(S\cup\{i\})}$, if player i did not deviate from x_i^σ and remained unconnected (waiting for his best individual opportunity or for his best collective opportunity), then the game would not end, because some player in $N \setminus (S \cup \{i\})$ would connect in subsequent stages. Indeed, the game would not end until this best individual or collective opportunity arrived, that is, at a certain stage, a set of nodes, $R_0 = R \cup \{0\} \in 2_0^N(i)$, $S_0 \subset R_0$, such that $B^i \cap R_0 \neq \emptyset$ or $i \in M^R$, would already be connected. Therefore, if player i had used an opportune moment strategy, x_i^σ, he could have connected at this stage paying a cost $c_i(x^\sigma) = m_i^R \leq m_i^S \leq c_{ij} = c_i(x^\sigma; x_i)$.

2. $B^i \cap S_0 = \emptyset$, $i \in M^S$ and $f_\sigma(M^S) = i$, but either
   (a) $(x^\sigma; x_i)_i(S_0) = x_i(S_0) = j \notin S_0^i$, or
   (b) $(x^\sigma; x_i)_i(S_0) = x_i(S_0) = a$.

   It is clear that player i does not improve by connecting to $j \notin S_0^i$. On the other hand, if player i deviates from x^σ by using x_i, which consists of remaining unconnected, and the rest of the agents play $x^{\sigma}_{-(S\cup\{i\})}$, even when $M_*^S \neq \emptyset$ and agents in M_*^S connect to S_0, then a cheaper connection for agent i will not appear, since agents in M_*^S connect at their cheapest connections which are not cheaper than m_i^S. Therefore, agent i does not improve by deviating from his opportune moment strategy and losing his best collective opportunity. Moreover, in the next stage of the game, when a set of nodes, $R_0 = R \cup \{0\} \in 2_0^N(i)$, $R_0 = S_0 \cup M_*^S$ is already connected, the situation is the same, that is, $B^i \cap R_0 = \emptyset$, $i \in M^R$ and $f_\sigma(M^R) = i$. Therefore, agent i does not improve by delaying his connection.

3. $B^i \cap S_0 \neq \emptyset$, but either
   (a) $(x^\sigma; x_i)_i(S_0) = x_i(S_0) = j \notin B^i \cap S_0$, or
   (b) $(x^\sigma; x_i)_i(S_0) = x_i(S_0) = a$.

   Since $B^i = \{j \in N_0 \setminus \{i\} \mid c_{ij} \leq c_{ik}, \forall k \in N_0, k \neq i\}$, then player i does not improve by joining $j \notin B^i \cap S$, nor by waiting for a cheaper connection and therefore $c_i(x^\sigma; x_i) \geq c_i(x^\sigma)$.

Conversely, let x be a profile of strategies for the set of players, N, that induce a MCST on N_0. Let T^x be the induced MCST. Since, by using Prim's algorithm, all minimum cost spanning trees in the graph can be generated, consider the ranking of the players, σ, that reflects how the players are connected by Prim's algorithm when T^x is the result. Consider also the profile of opportune moment strategies x^σ. Obviously, $c_i(x) \geq c_i(x^\sigma)$ for all $i \in N$. Moreover, the equality holds since if, at a stage when the nodes in $S_0 = S \cup \{0\}$ are connected, the players in M_*^S (if $M_*^S \neq \emptyset$) use their cheapest connection, although they are not due to connect at that stage, then a cheaper connection for any unconnected node, i, will not appear. Therefore, $c_i(x) = c_i(x^\sigma)$ for all players that, when using x^σ, do not use their best individual opportunity (and it is not their best collective opportunity). The equality holds also for those players that, by playing x^σ, use their best individual opportunity when it is not their best collective opportunity. This happens due to the fact that they use their cheapest connection which is already available. □

Note that in the converse part of Theorem 4.3 we have established that the allocation of costs provided by any profile of strategies inducing a MCST (which is not necessarily NE) can always be attained by a profile of opportune moment strategies (although perhaps the MCST obtained is not the same since a profile of opportune moment strategies and a profile of strategies based on Prim's algorithm as defined in (4.1) may induce more than a single MCST). The profile of strategies, y, for the game described in Example 3, induces a MCST, T', and it is not a NE. However, it is easy to see that by using a profile of opportune moment strategies, regardless of the ranking considered, the MCST, T', can be generated, and the players will pay the corresponding costs.

Another consequence of Theorem 4.3 that relates this non-cooperative solution to the cooperative approach is that, since opportune moment strategies generate minimum cost spanning trees, then the allocations of costs they induce coincide with those obtained by using Bird's rule, and therefore, are in the core of the cooperative minimum cost spanning tree game.

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